

EXPLICIT RESULTS CONCERNING QUANTUM QUASI-SHUFFLE ALGEBRAS AND THEIR APPLICATIONS

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ABSTRACT. Using the concept of mixable shuffles, we formulate explicitly the quantum quasi-shuffle product. We also provide a desirable description of the subalgebra generated by the set of primitive elements of the quantum quasi-shuffle bialgebra. A braided coalgebra structure which is dual to the quantum quasi-shuffle in some sense is constructed as well. We use quantum quasi-shuffle algebras to provide examples of Rota-Baxter algebras and tridendriform algebras.

1. INTRODUCTION

In [22], Ree introduced the shuffle algebra which has been studied extensively during the last fifty years. The shuffle product is carried out on the tensor space $T(V)$ of a vector space V by using the shuffle rule. Its natural generalization is the quasi-shuffle product where V is moreover an associative algebra and the new product on $T(V)$ involves both of the shuffle product and the multiplication of V . Quasi-shuffle algebras first arose in the work of Newman and Radford [20] for the study of cofree irreducible Hopf algebras built on associative algebras. Later, they were discussed by many other mathematicians for various motivations, such as multiple zeta values ([9] and [11]), Rota-Baxter algebras ([7],[6] and [4]), and commutative tridendriform algebras ([17]).

For both of physical and mathematical considerations, people want to deform or quantize some important algebra structures. The most famous example is absolutely the quantum group introduced by Drinfeld [3] and Jimbo [14]. To our surprise, there is an implicit but significant connection between quantum groups and shuffle algebras. Rosso [23] constructed the quantization of shuffle algebras. This is a new kind of quantized algebras and leads to an intrinsic understanding of the quantum group. Since shuffle algebras are special quasi-shuffle algebras, and the importance of the later ones, people would expect to find out what the quantization of quasi-shuffle algebras is and whether it can bring us some useful information. Hoffman's q -deformation of quasi-shuffles ([10]) is such an attempt. His idea is to deform the multiplication formula according to a special case of Rosso's quantum shuffles. After replacing the shuffle part by its quantized version in the formula of quasi-shuffles, Hoffman tried to deform the mixed term by multiplying a power of q , and found that there is only one way making the new product to be associative. This construction is more or less experiential as he said. The general construction

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of quantized quasi-shuffles is due to Rosso ([24]) in the spirit of his quantum shuffles. We describe Rosso's idea as follows. Let M be a Hopf bimodule over a Hopf algebra H . In addition, if M is an algebra and the multiplication is compatible with the braiding coming from the Hopf bimodule structure in some sense, then one can construct a new algebra structure on the cotensor coalgebra $T_H^c(M)$ by using its universal property. In general, given a braided algebra (A, m, σ) , one can construct an analogue of the quasi-shuffle algebra in the braided category, where the action of the usual flip is replaced by that of the braiding. The resulting algebra is called a quantum quasi-shuffle algebra. In particular, Hoffman's q -deformation of quasi-shuffle products is a special case of Rosso's quantum quasi-shuffle. In [12], the construction of quantum quasi-shuffles appears, as a special braided cofree Hopf algebra, in the framework of quantum multi-brace algebras.

Some interesting properties of the quantum quasi-shuffle algebras have been studied in [13], including the commutativity, universal property, and etc. This paper continues the trip. We establish some explicit results concerning this new subject. We start by reformulating the product. Originally, the quantum quasi-shuffle algebra is constructed by using the universal property of connected coalgebras ([12]). Later, it is defined by an inductive formula ([13]). But neither of these two constructions can provide an explicit formula. To know more about this new subject, a more clear form of the multiplication formula is definitely helpful. Here, we use the notion of mixable shuffles introduced in [7] to establish a complete description of the quantum quasi-shuffle product. Contrast to the quantum symmetric algebra, we describe the subalgebra of the quantum quasi-shuffle bialgebra generated by the primitive elements. On the other hand, for the reason that the universal property of connected coalgebras is not so familiar by non-algebraists, we use the universal property of tensor algebras to construct a braided coalgebra structure on $T(C)$ for a braided coalgebra C , and show that its dual is the quantum quasi-shuffle algebra. This enables one to study the quantum quasi-shuffle algebra through its dual. Finally, as applications, we provide examples of Rota-Baxter algebras and tridendriform algebras by using quantum quasi-shuffles. Rota-Baxter algebras and tridendriform algebras are important subjects in mathematics and physics. So our constructions not only enlarge their families, but also demonstrate the value of the quantum quasi-shuffles.

This paper is organized as follows. In Section 2, several concrete examples of braided algebras are provided. In Section 3, we recall the construction of quantum quasi-shuffle algebras and established explicit formulas for the new product and the subalgebra generated by the primitive elements. In Section 4, we construct the dual coalgebra of the quantum quasi-shuffle algebra. Finally, in Section 5, it contains some applications of the quantum quasi-shuffle algebra involving examples of Rota-Baxter algebras and tridendriform algebras.

Notation. In this paper, we denote by \mathbb{K} a ground field of characteristic 0. All the objects we discuss are defined over \mathbb{K} . For a vector space V , we denote by \otimes the tensor product within $T(V)$, and by $\underline{\otimes}$ the one between $T(V)$ and $T(V)$.

We denote by \mathfrak{S}_n the symmetric group of the set $\{1, 2, \dots, n\}$ and by s_i , $1 \leq i \leq n-1$, the standard generators of \mathfrak{S}_n permuting i and $i+1$. For fixed $k, n \in \mathbb{N}$, we define the shift map $\text{shift}_k : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n+k}$ by $\text{shift}_k(s_i) = s_{i+k}$ for any $1 \leq i \leq n-1$. For the reason of intuition and the simplicity of notation, we denote $1_{\mathfrak{S}_k} \times w =$

$\text{shift}_k(w)$ for any $w \in \mathfrak{S}_n$. The notations $w \times 1_{\mathfrak{S}_k}$, $1_{\mathfrak{S}_k} \times w \times 1_{\mathfrak{S}_l}$ and others are understood similarly.

A braiding σ on a vector space V is an invertible linear map in $\text{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation on $V^{\otimes 3}$:

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space (V, σ) is a vector space V equipped with a braiding σ . For any $n \in \mathbb{N}$ and $1 \leq i \leq n-1$, we denote by σ_i the operator $\text{id}_V^{\otimes i-1} \otimes \sigma \otimes \text{id}_V^{\otimes n-i-1} \in \text{End}(V^{\otimes n})$. For any $w \in \mathfrak{S}_n$, we denote by T_w^σ the corresponding lift of w in the braid group B_n , defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of w , then $T_w^\sigma = \sigma_{i_1} \cdots \sigma_{i_l}$. This definition is well-defined (see, e.g., Theorem 4.12 in [16]).

We define $\beta : T(V) \underline{\otimes} T(V) \rightarrow T(V) \underline{\otimes} T(V)$ by requiring that the restriction of β on $V^{\otimes i} \underline{\otimes} V^{\otimes j}$, denoted by β_{ij} , is $T_{\chi_{ij}}^\sigma$, where

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix} \in \mathfrak{S}_{i+j},$$

for any $i, j \geq 1$. For convenience, we denote by β_{0i} and β_{i0} the usual flip.

2. BRAIDED ALGEBRAS

We start by recalling the notion of braided algebras which is the correspondent object of associative algebras in braided categories. In the following, all algebras are assumed to be associative but not necessarily unital.

Definition 2.1. Let $A = (A, m)$ be an algebra with product m , and σ be a braiding on A . We call the triple (A, m, σ) a *braided algebra* if it satisfies the following conditions:

$$\begin{cases} (\text{id}_A \otimes m)\sigma_1\sigma_2 &= \sigma(m \otimes \text{id}_A), \\ (m \otimes \text{id}_A)\sigma_2\sigma_1 &= \sigma(\text{id}_A \otimes m). \end{cases}$$

Moreover, if A is unital and its unit 1_A satisfies that for any $a \in A$,

$$\begin{cases} \sigma(a \otimes 1_A) &= 1_A \otimes a, \\ \sigma(1_A \otimes a) &= a \otimes 1_A, \end{cases}$$

then A is called a *unital braided algebra*.

Remark 2.2. 1. For any braided vector space (V, σ) , there is a trivial braided algebra structure on it whose multiplication is the trivial one $m = 0$.

2. The braided algebra structure is very crucial. Given a braided vector space (V, σ) , it is not reasonable that there should be a non-trivial braided algebra structure on V . For instance, let V be a vector space with basis $\{e_1, e_2\}$. We define two braidings σ_1 and σ_2 on V respectively by

$$\begin{cases} \sigma_1(e_1 \otimes e_1) &= e_1 \otimes e_1, \\ \sigma_1(e_i \otimes e_2) &= qe_2 \otimes e_1, \\ \sigma_1(e_2 \otimes e_1) &= qe_1 \otimes e_2 + (1 - q^2)e_2 \otimes e_1, \\ \sigma_1(e_2 \otimes e_2) &= e_2 \otimes e_2, \end{cases}$$

and

$$\sigma_2(e_i \otimes e_j) = qe_j \otimes e_i, \quad \forall i, j,$$

where $q \in \mathbb{K}$ is nonzero and not equal to ± 1 .

Then by an easy argument, one can show that the only product on V which is compatible with σ_1 is just the trivial one. The case is the same for σ_2 .

3. For any braided algebra (A, m, σ) , one can embed it into a unital braided algebra $(\tilde{A}, \tilde{m}, \tilde{\sigma})$ in the following way. First of all, we set $\tilde{A} = \mathbb{K} \oplus A$. Then we define the multiplication \tilde{m} and the braiding $\tilde{\sigma}$ by: for any $\lambda, \mu \in \mathbb{K}$ and $a, b \in A$

$$\tilde{m}((\lambda + a) \otimes (\mu + b)) = \lambda\mu + \lambda \cdot b + \mu \cdot a + m(a \otimes b),$$

and

$$\tilde{\sigma}((\lambda + a) \otimes (\mu + b)) = \mu \otimes \lambda + b \otimes \lambda + \mu \otimes a + \sigma(a \otimes b).$$

It is easy to verify that $(\tilde{A}, \tilde{m}, \tilde{\sigma})$ is a braided algebra with unit $1 \in \mathbb{K}$.

Because all the constructions in this paper are based on braided algebras, we provide several concrete examples which will either be used in our later discussion or afford the reader some illustrations. Some of them may be known, while some may be new. For more examples, one can see [1] and [12].

Example 2.3. Let V be a vector space with basis $\{e_i\}$ which is at most countable. We provide a braided algebra structure on V . The braiding σ on V is given by $\sigma(e_i \otimes e_j) = q_{ij}e_j \otimes e_i$, where q_{ij} 's are nonzero scalars in \mathbb{K} such that $q_{ij}q_{ik} = q_{i+j+k}$ and $q_{ik}q_{jk} = q_{i+j+k}$ for any i, j, k . For instance, let q be a nonzero scalar in \mathbb{K} and $q_{ij} = q^{ij}$. The multiplication \cdot on V which is compatible with the braiding σ is given as follows.

Case 1. If V is a finite-dimensional vector space with basis $\{e_1, e_2, \dots, e_N\}$, then we define

$$e_i \cdot e_j = \begin{cases} e_{i+j}, & \text{if } i+j \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Case 2. If V is a vector space with basis $\{e_i\}_{i \in \mathbb{N}}$, then we define $e_i \cdot e_j = e_{i+j}$ for any $i, j \in \mathbb{N}$.

It is evident that \cdot is an associative algebra structure on V in both cases. Notice that

$$\begin{aligned} (\text{id}_V \otimes \cdot)\sigma_1\sigma_2(e_i \otimes e_j \otimes e_k) &= q_{jk}q_{ik}e_k \otimes e_{i+j} \\ &= q_{i+j+k}e_k \otimes e_{i+j} \\ &= \sigma(\cdot \otimes \text{id}_V)(e_i \otimes e_j \otimes e_k), \end{aligned}$$

and similarly $(\cdot \otimes \text{id}_V)\sigma_2\sigma_1 = \sigma(\text{id}_V \otimes \cdot)$. Therefore (V, \cdot, σ) is a braided algebra.

Example 2.4. All notions of this example can be found in [15]. Let $q \neq 1$ be a invertible scalar in \mathbb{K} , and x, y be two indeterminates. Denote by $\mathbb{K}_q[x, y]$ the quantum plane, i.e., the algebra generated by x, y with the relation $yx = qxy$. It

has a linear basis $\{x^i y^j\}_{i,j \geq 0}$. Define two algebra automorphisms ω_x and ω_y of $\mathbb{K}_q[x, y]$ by requiring that

$$\omega_x(x) = qx, \omega_x(y) = x, \omega_y(x) = x, \omega_y(y) = qy,$$

and define two endomorphisms $\partial_q/\partial x$ and $\partial_q/\partial y$ by requiring that

$$\frac{\partial_q(x^m y^n)}{\partial x} = [m]x^{m-1}y^n, \frac{\partial_q(x^m y^n)}{\partial y} = x^m[n]y^{n-1},$$

where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ for any $k \in \mathbb{N}$.

Let $U_q \mathfrak{sl}_2$ be the quantized algebra associated to \mathfrak{sl}_2 , i.e., the algebra generated by E, F, K, K^{-1} with the relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KE &= q^2 EK, \quad KF = q^{-2} FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

It is well-known that $U_q \mathfrak{sl}_2$ is a quasi-triangular Hopf algebra.

By Theorem VII 3.3 in [15], $\mathbb{K}_q[x, y]$ is a $U_q \mathfrak{sl}_2$ -module-algebra with the following module structure: for any $P \in \mathbb{K}_q[x, y]$,

$$EP = x \frac{\partial_q(P)}{\partial y}, \quad FP = \frac{\partial_q(P)}{\partial x} y,$$

$$KP = (\omega_x \omega_y^{-1})(P), \quad K^{-1}P = (\omega_y \omega_x^{-1})(P).$$

Set $V = \text{Span}_{\mathbb{K}}\{x, y\}$. It is not hard to see that the above action restricting on V is the standard 2-dimensional simple $U_q \mathfrak{sl}_2$ -module structure. We know that (Theorem 2.7 in [12]) every module-algebra over a quasi-triangular Hopf algebra has a braided algebra structure. So $\mathbb{K}_q[x, y]$ is a braided algebra.

Example 2.5. Let (V, σ) be a braided vector space. It is known that $(T(V), m, \beta)$ is a braided algebra, where m is the concatenation product. Let $M_n : V^{\otimes n} \rightarrow T(V)$ be a linear map such that $\beta(M_n \otimes \text{id}_V) = (\text{id}_V \otimes M_n)\beta_{n1}$. If we denote by \mathcal{I} the ideal of $T(V)$ generated by $\text{Im} M_n$, the image of M_n , then $\beta(T(V) \underline{\otimes} \mathcal{I} + \mathcal{I} \underline{\otimes} T(V)) \subset T(V) \underline{\otimes} \mathcal{I} + \mathcal{I} \underline{\otimes} T(V)$. So the quotient algebra $T(V)/\mathcal{I}$ is also a braided algebra. For instance, if $M_2 = \text{id}_V^{\otimes 2} - \sigma$, then the quotient algebra is the r -symmetric algebra defined in [1].

Example 2.6. Let H be a finite dimensional quasi-triangular Hopf algebra. By a result of Majid (Theorem 3.3 in [19]), the quantum double $\mathcal{D}(H)$ of H is a braided algebra (according to a discussion in [12] for Radford's work [21]).

3. QUANTUM QUASI-SHUFFLE ALGEBRAS

For any algebra A , it is a braided algebra with respect to the flip map switching the two factors of $A \otimes A$. One can construct an algebra structure on $T(A)$ which combines the multiplication of A and the shuffle product of $T(A)$ (see [20]). This structure is the so-called quasi-shuffle algebra. In fact, if the flip map is replaced by a braiding, one can construct a quantized quasi-shuffle product by assuming some compatibilities between the multiplication of A and the braiding (for more details, one can see [12] and [13]). Given a braided algebra (A, m, σ) ,

the *quantum quasi-shuffle product* \bowtie_σ on $T(A)$ is given by the following inductive formula: for $i, j > 1$ and any $a_1, \dots, a_i, b_1, \dots, b_j \in A$,

$$\begin{aligned} & (a_1 \otimes \cdots \otimes a_i) \bowtie_\sigma (b_1 \otimes \cdots \otimes b_j) \\ &= a_1 \otimes ((a_2 \otimes \cdots \otimes a_i) \bowtie_\sigma (b_1 \otimes \cdots \otimes b_j)) \\ & \quad + (\text{id}_A \otimes \bowtie_{\sigma(i, j-1)})(\beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j) \\ & \quad + (m \otimes \bowtie_{\sigma(i-1, j-1)})(\text{id}_A \otimes \beta_{i-1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j), \end{aligned}$$

where $\bowtie_{\sigma(i,j)}$ denotes the restriction of \bowtie_σ on $V^{\otimes i} \otimes V^{\otimes j}$.

Remark 3.1. 1. Given a braided algebra (A, m, σ) , $T_\sigma^{qsh}(A) = (T(A), \bowtie_\sigma)$ is an associative algebra with unit $1 \in \mathbb{K}$, and called the *quantum quasi-shuffle algebra* built on (A, m, σ) . Furthermore, the algebra $T_\sigma^{qsh}(A)$, together with the braiding β and the deconcatenation coproduct δ , forms a braided bialgebra in the sense of [25] (see [12]). The set of primitive elements is exactly A .

2. By using Example 2.3, Proposition 17 in [13] can be applied to any vector space whose basis is at most countable. In other words, for any vector space V with at most countable basis, one can provide a linear basis of $T(V)$ by combining the quantum quasi-shuffle product with Lyndon words.

Example 3.2 (Hoffman's q-deformation). In [10], Hoffman defined his q-deformation of quasi-shuffles. It is an attempt to deform the quasi-shuffle product according to the quantum shuffle product. Now we give an explanation of the q-deformation from a point of view of the quantum quasi-shuffles. Let X be a locally finite set, i.e., X is a disjoint union of finite set X_n , whose elements are called letters of degree n , for $n \geq 1$. We denote by \mathfrak{X} the vector space spanned by X . The elements in $T(\mathfrak{X})$, which are of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_m$ with $a_i \in X$, are called words. Let $[\cdot, \cdot]$ be a graded associative product on \mathfrak{X} . Hoffman defined an associative product $*_q$ on $T(\mathfrak{X})$: for any words $w_1, w_2 \in T(\mathfrak{X})$ and letters $a, b \in X$,

$$\begin{aligned} & (a \otimes w_1) *_q (b \otimes w_2) \\ &= a \otimes (w_1 *_q (b \otimes w_2)) + q^{|a \otimes w_1| |b|} b \otimes ((a \otimes w_1) *_q w_2) + q^{|w_1| |b|} [a, b] (w_1 *_q w_2), \end{aligned}$$

where $|w|$ is the degree of a word, i.e., the sum of degrees of its factors.

We define a braiding σ on \mathfrak{X} as follows: for $x \in X_i$ and $y \in X_j$, $\sigma(x \otimes y) = q^{ij} y \otimes x$, where $q \in K$ is a nonzero scalar. Since the product $[\cdot, \cdot]$ preserves the grading, it is an easy exercise to verify that $(\mathfrak{X}, [\cdot, \cdot], \sigma)$ is a braided algebra. By comparing their reductive formulas, one can see that the quantum quasi-shuffle algebra built on $(\mathfrak{X}, [\cdot, \cdot], \sigma)$ is just Hoffman's q-deformation.

In order to give a more explicit description of the quantum quasi-shuffle product, we need to recall some terminologies introduced in [7]. An (i, j) -shuffle is an element $w \in \mathfrak{S}_{i+j}$ such that $w(1) < \cdots < w(i)$ and $w(i+1) < \cdots < w(i+j)$. We denote by $\mathfrak{S}_{i,j}$ the set of all (i, j) -shuffles. Given an (i, j) -shuffle w , a pair $(k, k+1)$, where $1 \leq k < i+j$, is called an *admissible pair* for w if $w^{-1}(k) \leq i < w^{-1}(k+1)$. We denote by \mathcal{T}^w the set of all admissible pairs for w . For any subset S of \mathcal{T}^w , the pair (w, S) is called a *mixable (i, j) -shuffle*. We denote by $\overline{\mathfrak{S}}_{i,j}$ the set of all mixable (i, j) -shuffles, i.e.,

$$\overline{\mathfrak{S}}_{i,j} = \{(w, S) | w \in \mathfrak{S}_{i,j}, S \subset \mathcal{T}^w\}.$$

Let (A, m, σ) be a braided algebra. Define $m^k : A^{\otimes k+1} \rightarrow A$ recursively by $m^0 = \text{id}_A$, $m^1 = m$ and $m^k = m(\text{id}_A \otimes m^{k-1})$ for $k \geq 2$. Given $n \in \mathbb{N}$, we denote

$$\mathcal{C}(n) = \{I = (i_1, \dots, i_k) \in \mathbb{N}^k \mid i_1 + \dots + i_k = n\}.$$

The elements in $\mathcal{C}(n)$ are called *compositions* of n . For any $I = (i_1, \dots, i_k) \in \mathcal{C}(n)$, we define $m_I = m^{i_1-1} \otimes \dots \otimes m^{i_k-1}$. For any $(w, S) \in \overline{\mathfrak{S}}_{i,j}$, we associate to S a composition $\text{cp}(S)$ of $i+j$ as follows: if $S = \{(k_1, k_1+1), \dots, (k_s, k_s+1)\}$ with $k_1 < \dots < k_s$, set

$$\text{cp}(S) = (\underbrace{1, \dots, 1}_{k_1-1 \text{ copies}}, 2, \underbrace{1, \dots, 1}_{k_2-k_1-2 \text{ copies}}, 2, \dots, 2, \underbrace{1, \dots, 1}_{i+j-k_s-1 \text{ copies}}).$$

By convention, we set $\text{cp}(\emptyset) = (1, 1, \dots, 1)$. Denote $T_{(w,S)}^\sigma = m_{\text{cp}(S)} \circ T_w^\sigma$.

Theorem 3.3. *Let (A, m, σ) be a braided algebra. Then for any $a_1, \dots, a_{i+j} \in A$,*

$$(a_1 \otimes \dots \otimes a_i) \bowtie_\sigma (a_{i+1} \otimes \dots \otimes a_{i+j}) = \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j}} T_{(w,S)}^\sigma(a_1 \otimes \dots \otimes a_{i+j}).$$

Proof. We use induction on $i+j$.

When $i = j = 1$,

$$a_1 \bowtie_\sigma a_2 = m(a_1 \otimes a_2) + a_1 \otimes a_2 + \sigma(a_1 \otimes a_2).$$

On the other hand, $\overline{\mathfrak{S}}_{1,1} = \{(1_{\mathfrak{S}_2}, \emptyset), (1_{\mathfrak{S}_2}, \{(1, 2)\}), (s_1, \emptyset)\}$, where s_1 is the generator of \mathfrak{S}_2 . So the formula holds.

We assume that the formula is true in the case $\leq i+j$. By the inductive formula of quantum quasi-shuffles, we have that

$$\begin{aligned} & (a_1 \otimes \dots \otimes a_{i+1}) \bowtie_\sigma (a_{i+2} \otimes \dots \otimes a_{i+j+1}) \\ &= a_1 \otimes ((a_2 \otimes \dots \otimes a_{i+1}) \bowtie_\sigma (a_{i+2} \otimes \dots \otimes a_{i+j+1})) \\ & \quad + (\text{id}_A \otimes \bowtie_{\sigma(i+1,j-1)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_{i+j+1}) \\ & \quad + (m \otimes \bowtie_{\sigma(i,j-1)})(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_{i+j+1}) \\ &= \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j}} (\text{id}_A \otimes T_{(w,S)}^\sigma)(a_1 \otimes \dots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_{i+1,j-1}} (\text{id}_A \otimes T_{(w,S)}^\sigma)(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j-1}} (m \otimes T_{(w,S)}^\sigma)(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a_1 \otimes \dots \otimes a_{i+j+1}) \\ &= \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j}} (\text{id}_A \otimes m_{\text{cp}(S)})T_{1_{\mathfrak{S}_1} \times w}^\sigma(a_1 \otimes \dots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_{i+1,j-1}} (\text{id}_A \otimes m_{\text{cp}(S)})T_{(1_{\mathfrak{S}_1} \times w) \circ (\chi_{i+1,1} \times 1_{\mathfrak{S}_{j-1}})}^\sigma(a_1 \otimes \dots \otimes a_{i+j+1}) \\ & \quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_{i,j-1}} (m \otimes m_{\text{cp}(S)})T_{(1_{\mathfrak{S}_2} \times w) \circ (1_{\mathfrak{S}_1} \times \chi_{i,1} \times 1_{\mathfrak{S}_{j-1}})}^\sigma(a_1 \otimes \dots \otimes a_{i+j+1}), \end{aligned}$$

where the third equality follows from the fact that all the expressions of the permutations being lifted are reduced.

Denote

$$\begin{aligned}\mathcal{S}_1 &= \{(w, S) \in \overline{\mathfrak{S}}_{i+1,j} | (1, 2) \notin S, w(1) = 1\}, \\ \mathcal{S}_2 &= \{(w, S) \in \overline{\mathfrak{S}}_{i+1,j} | (1, 2) \notin S, w(i+2) = 1\},\end{aligned}$$

and

$$\mathcal{S}_3 = \{(w, S) \in \overline{\mathfrak{S}}_{i+1,j} | (1, 2) \in S\}.$$

For any $(i+1, j)$ -shuffle w , one has either $w(1) = 1$ or $w(i+2) = 1$. Therefore \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 are mutually disjoint, and $\mathfrak{S}_{i+1,j} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

We make a further observation. It is easy to see that there is a one-to-one correspondence between $\mathfrak{S}_{i,j}$ and $\{w \in \mathfrak{S}_{i+1,j} | w(1) = 1\}$ given by $w \mapsto 1_{\mathfrak{S}_1} \times w$ for any $w \in \mathfrak{S}_{i,j}$. So

$$\mathcal{S}_1 = \{(1_{\mathfrak{S}_1} \times w, S) | w \in \mathfrak{S}_{i,j}, S \subset \mathcal{T}^{1_{\mathfrak{S}_1} \times w}, (1, 2) \notin S\}.$$

There is a one-to-one correspondence between $\mathfrak{S}_{i+1,j-1}$ and $\{w \in \mathfrak{S}_{i+1,j} | w(i+2) = 1\}$ given by $w \mapsto \tilde{w} = (1_{\mathfrak{S}_1} \times w) \circ (\chi_{i+1,1} \times 1_{\mathfrak{S}_{j-1}})$ for any $w \in \mathfrak{S}_{i+1,j-1}$. Consequently,

$$\mathcal{S}_2 = \{(\tilde{w}, S) | w \in \mathfrak{S}_{i+1,j-1}, S \subset \mathcal{T}^{\tilde{w}}, (1, 2) \notin S\}.$$

Finally, for any $(w, S) \in \mathcal{S}_3$, we must have that $w(1) = 1$ and $w(i+2) = 2$. There is a one-to-one correspondence between $\mathfrak{S}_{i,j-1}$ and $\{w \in \mathfrak{S}_{i+1,j} | w(1) = 1, w(i+2) = 2\}$ given by $w \mapsto \overline{w} = (1_{\mathfrak{S}_2} \times w) \circ (1_{\mathfrak{S}_1} \times \chi_{i,1} \times 1_{\mathfrak{S}_{j-1}})$ for any $w \in \mathfrak{S}_{i,j-1}$. So

$$\mathcal{S}_3 = \{(\overline{w}, S) \in \overline{\mathfrak{S}}_{i+1,j} | w \in \mathfrak{S}_{i,j-1}, (1, 2) \in S\}.$$

As a conclusion, the three terms in the final step of the preceding computation come from \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 respectively. So we have that

$$(a_1 \otimes \cdots \otimes a_{i+1}) \rtimes_{\sigma} (a_{i+2} \otimes \cdots \otimes v_{i+j+1}) = \sum_{(w,S) \in \overline{\mathfrak{S}}_{i+1,j}} T_{(w,S)}^{\sigma}(a_1 \otimes \cdots \otimes a_{i+j+1}),$$

which completes the induction. \square

Remark 3.4. Let (A, m) be an algebra and λ be a scalar in \mathbb{K} . Then $(A, \lambda m)$ becomes a braided algebra with respect to the usual flip map. In this case, the formula in Theorem 3.3 coincides with the one of mixable shuffle product introduced in [7].

Assume again that (A, m, σ) is a braided algebra. We denote by $S_{\sigma}^{qsh}(A)$ the subalgebra of $T_{\sigma}^{qsh}(A)$ generated by A . To describe this subalgebra, we need to introduce some notations.

For a fixed $n \in \mathbb{N}$ and any $w \in \mathfrak{S}_n$, we denote

$$\mathcal{S}^w = \{(k, k+1) | 1 \leq k < n, w^{-1}(k) < w^{-1}(k+1)\},$$

and

$$\overline{\mathfrak{S}}_n = \{(w, S) | w \in \mathfrak{S}_n, S \subset \mathcal{S}^w\}.$$

For any $(w, S) \in \overline{\mathfrak{S}}_n$, we associate to S a composition $\text{cp}(S)$ of n as follows. Let $S = \{(k_1, k_1 + 1), \dots, (k_s, k_s + 1)\}$ with $k_1 < \dots < k_s$. We divide $\{k_1, \dots, k_s\}$ into several subsets

$$\{k_1, \dots, k_{i_1}\}, \{k_{i_1+1}, \dots, k_{i_1+i_2}\}, \dots, \{k_{i_1+\dots+i_{r-1}+1}, \dots, k_s\},$$

which obey the rule that:

$$\begin{cases} k_1 + 1 = k_2, k_2 + 1 = k_3, \dots, k_{i_1-1} + 1 = k_{i_1}, \\ k_{i_1+1} + 1 = k_{i_1+2}, k_{i_1+2} + 1 = k_{i_1+3}, \dots, k_{i_1+i_2-1} + 1 = k_{i_2}, \\ \dots, \\ k_{i_1+\dots+i_{r-1}+1} + 1 = k_{i_1+\dots+i_{r-1}+2}, \dots, k_{s-1} + 1 = k_s, \end{cases}$$

but $k_{i_1} + 1 < k_{i_1+1}, k_{i_1+i_2} + 1 < k_{i_1+i_2+1}, \dots, k_{i_1+\dots+i_{r-1}} + 1 < k_{i_1+\dots+i_{r-1}+1}$. Denote $i_r = s - i_1 - \dots - i_{r-1}$. Then we write

$$\text{cp}(S) = (\underbrace{1, \dots, 1}_{k_1-1 \text{ copies}}, i_1 + 1, \underbrace{1, \dots, 1}_{k_{i_1+1}-k_{i_1}-1 \text{ copies}}, i_2 + 1, \dots, i_r + 1, \underbrace{1, \dots, 1}_{n-k_{i_r}-1 \text{ copies}}).$$

We define as before the map $T_{(w,S)}^\sigma = m_{\text{cp}(S)} \circ T_w^\sigma$ for any $(w, S) \in \overline{\mathfrak{S}}_n$.

Now we provide a decomposition of $\overline{\mathfrak{S}}_{n+1}$ which will be used later. For any $1 \leq i \leq n+1$, we denote $\mathfrak{S}_{n+1}(i) = \{w \in \mathfrak{S}_{n+1} | w(1) = i\}$. It is clear that \mathfrak{S}_{n+1} is the disjoint union of all $\mathfrak{S}_{n+1}(i)$'s, and for each i there is a one-to-one correspondence between \mathfrak{S}_n and $\mathfrak{S}_{n+1}(i)$ given by $w \mapsto L(w, i) = (\chi_{1,i-1} \times 1_{\mathfrak{S}_{n+1-i}}) \circ (1_{\mathfrak{S}_1} \times w)$ for any $w \in \mathfrak{S}_{n-1}$. So

$$\begin{aligned} \mathfrak{S}_{n+1} &= \bigcup_{i=1}^{n+1} \mathfrak{S}_{n+1}(i) \\ &= \bigcup_{i=1}^{n+1} \bigcup_{w \in \mathfrak{S}_n} \{L(w, i)\} \\ &= \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{L(w, i)\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \overline{\mathfrak{S}}_{n+1} &= \bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w,i)}\} \\ &= \left(\bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w,i)}, (i, i+1) \notin S\} \right) \\ &\quad \cup \left(\bigcup_{w \in \mathfrak{S}_n} \bigcup_{i=1}^{n+1} \{(L(w, i), S) | S \subset \mathcal{S}^{L(w,i)}, (i, i+1) \in S\} \right). \end{aligned}$$

All the unions above are disjoint.

Given $w \in \mathfrak{S}_n$ and $S \subset \mathcal{S}^w$ with $\text{cp}(S) = (i_1, \dots, i_s)$. For any $0 \leq k \leq s$, we denote

$$\begin{aligned} \text{cp}(S)_k &= (i_1, \dots, i_k, 1, i_{k+1}, \dots, i_s), \\ \text{cp}(S)^k &= (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_s), \end{aligned}$$

and

$$I_k = (\underbrace{1, \dots, 1}_k, 2, \underbrace{1, \dots, 1}_{n-1-|S|-k}).$$

Here, $|S|$ denotes the cardinality of the set S .

Lemma 3.5. *Under the assumptions above, we have*

$$\begin{cases} (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) &= m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}, \\ m_{I_k}(\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) &= m_{\text{cp}(S)^k} T_{L(w, i_1 + \dots + i_k + 1)}. \end{cases}$$

Proof. Since $\sigma(\text{id}_A \otimes m^l) = (m^l \otimes \text{id}_A)\beta_{1,l+1}$ for any l (see Lemma 2 in [1]),

$$\begin{aligned} & (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes T_{(w,S)}^\sigma) \\ &= (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k})(\text{id}_A \otimes m_{\text{cp}(S)})(\text{id}_A \otimes T_w^\sigma) \\ &= m_{\text{cp}(S)_k}(\beta_{1, i_1 + \dots + i_k} \otimes \text{id}_A^{\otimes i_{k+1} + \dots + i_s})(\text{id}_A \otimes T_w^\sigma) \\ &= m_{\text{cp}(S)_k} T_{(\chi_{1, i_1 + \dots + i_k} \times 1_{\mathfrak{S}_{i_{k+1} + \dots + i_s}}) \circ (1_{\mathfrak{S}_1} \times w)}^\sigma \\ &= m_{\text{cp}(S)_k} T_{L(w, i_1 + \dots + i_k + 1)}. \end{aligned}$$

The second equality is a consequence of the first one. \square

Theorem 3.6. *Let (A, m, σ) be a braided algebra. For any $a_1, \dots, a_n \in A$, we have that*

$$a_1 \bowtie_\sigma \dots \bowtie_\sigma a_n = \sum_{(w,S) \in \overline{\mathfrak{S}}_n} T_{(w,S)}^\sigma(a_1 \otimes \dots \otimes a_n).$$

Therefore $S_\sigma^{qsh}(A) = \sum_{n \geq 0} \text{Im}(\sum_{(w,S) \in \overline{\mathfrak{S}}_n} T_{(w,S)}^\sigma)$.

Proof. We use induction on n .

When $n = 2$, it is trivial since

$$\overline{\mathfrak{S}}_2 = \{(1_{\mathfrak{S}_2}, \emptyset), (1_{\mathfrak{S}_2}, \{(1, 2)\}), (s_1, \emptyset)\}.$$

By Theorem 3.5, we have that for any $a_1, \dots, a_{r+1} \in A$,

$$\begin{aligned} & a_1 \bowtie_\sigma (a_2 \otimes \dots \otimes a_{r+1}) \\ &= \sum_{k=0}^r (\beta_{1,k} \otimes \text{id}_A^{\otimes r-k})(a_1 \otimes \dots \otimes a_{r+1}) \\ & \quad + \sum_{k=0}^{r-1} (\text{id}_A^{\otimes k} \otimes m \otimes \text{id}_A^{\otimes r-k-1})(\beta_{1,k} \otimes \text{id}_A^{\otimes r-k})(a_1 \otimes \dots \otimes a_{r+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & a_1 \bowtie_\sigma \dots \bowtie_\sigma a_{n+1} \\ &= a_1 \bowtie_\sigma \left(\sum_{(w,S) \in \overline{\mathfrak{S}}_n} T_{(w,S)}^\sigma(a_2 \otimes \dots \otimes a_{n+1}) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{(w,S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|} (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k}) (a_1 \otimes T_{(w,S)}^\sigma(a_2 \otimes \cdots \otimes a_{n+1})) \\
&\quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|-1} (\text{id}_A^{\otimes k} \otimes m \otimes \text{id}_A^{\otimes n-|S|-k-1}) \\
&\quad \quad \quad \circ (\beta_{1,k} \otimes \text{id}_A^{\otimes n-|S|-k}) (a_1 \otimes T_{(w,S)}^\sigma(a_2 \otimes \cdots \otimes a_{n+1})) \\
&= \sum_{(w,S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|} m_{\text{cp}(S)_k} T_{L(w,i_1+\dots+i_k+1)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad + \sum_{(w,S) \in \overline{\mathfrak{S}}_n} \sum_{k=0}^{n-|S|-1} m_{\text{cp}(S)_k} T_{L(w,i_1+\dots+i_k+1)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}),
\end{aligned}$$

where the last equality follows from the preceding lemma.

On the other hand, by the decomposition of $\overline{\mathfrak{S}}_{n+1}$ mentioned before,

$$\begin{aligned}
&\sum_{(w,S) \in \overline{\mathfrak{S}}_{n+1}} T_{(w,S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}) \\
&= \sum_{w \in \mathfrak{S}_n} \sum_{i=1}^{n+1} \sum_{\substack{S \subset \mathcal{S}^{L(w,i)} \\ (i,i+1) \notin S}} T_{(L(w,i),S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad + \sum_{w \in \mathfrak{S}_n} \sum_{i=1}^{n+1} \sum_{\substack{S \subset \mathcal{S}^{L(w,i)} \\ (i,i+1) \in S}} T_{(L(w,i),S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

We compare the terms in these two expressions. Notice that for a fixed $1 \leq i \leq n+1$ and $S \subset \mathcal{S}^{L(w,i)}$ with $(i,i+1) \notin S$, there is a unique $S' \in \mathcal{S}^w$ such that $\text{cp}(S) = \text{cp}(S')_i$. Indeed, we can write down S' explicitly: if $S = \{(k_1, k_1+1), \dots, (k_s, k_s+1)\}$ with $k_1 < \dots < k_l < i < k_{l+1} < \dots < k_s$, then $S' = \{(k_1, k_1+1), \dots, (k_l, k_l+1), (k_{l+1}-1, k_{l+1}), \dots, (k_s-1, k_s)\}$. Similarly, if $(i,i+1) \in S$, there is a unique $S'' \in \mathcal{S}^w$ such that $\text{cp}(S) = \text{cp}(S'')^i$. It follows that every term in $\sum_{(w,S) \in \overline{\mathfrak{S}}_{n+1}} T_{(w,S)}^\sigma(a_1 \otimes \cdots \otimes a_{n+1})$ is from exactly one term in the formula of $a_1 \bowtie_\sigma \cdots \bowtie_\sigma a_{n+1}$. The converse is also true. Since all terms in each formula are mutually distinct, we get the conclusion. \square

Remark 3.7. Consider Example 3.2 and let $(\mathfrak{X}, [\cdot, \cdot], \sigma)$ be the braided algebra introduced there. For any $w \in \mathfrak{S}_n$, we denote $\iota(w) = \{(i,j) | 1 \leq i < j \leq n, w(i) > w(j)\}$. Then for any $a_1, \dots, a_n \in X$,

$$T_w^\sigma(a_1 \otimes \cdots \otimes a_n) = q^{\sum_{(i,j) \in \iota(w)} |a_i| |a_j|} a_{w^{-1}(1)} \otimes \cdots \otimes a_{w^{-1}(n)}.$$

For any two compositions $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ of n , we say I is a *refinement* of J , written by $I \succeq J$, if there are $r_1, \dots, r_l \in \mathbb{N}$ such that $r_1 + \dots + r_l = k$ and

$$i_1 + \cdots + i_{r_1} = j_1, i_{r_1+1} + \cdots + i_{r_1+r_2} = j_2, \dots, i_{r_1+\dots+r_{l-1}+1} + \cdots + i_k = j_l.$$

For instance, $(1, 2, 2, 3) \succeq (3, 2, 3)$. For any $w \in \mathfrak{S}_n$, let $C(w)$ be the composition (i_1, \dots, i_k) of n such that

$$\{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{k-1}\} = \{l | 1 \leq l \leq n-1, w(l) > w(l+1)\}.$$

For any $I = (i_1, \dots, i_l) \in \mathcal{C}(n)$, we write $I[a_1 \otimes \dots \otimes a_n] = [\cdot]_I(a_1 \otimes \dots \otimes a_n)$.

Observing that for any $w \in \mathfrak{S}_n$ there is a one-to-one correspondence between $S \subset \mathcal{S}^w$ and $I \in \mathcal{C}(n)$ with $I \succeq C(w)$, one has immediately that

$$a_1 *_q \dots *_q a_n = \sum_{w \in \mathfrak{S}_n} q^{\sum_{(i,j) \in \iota(w)} |a_i| |a_j|} \sum_{I \succeq C(w)} I[a_{w^{-1}(1)} \otimes \dots \otimes a_{w^{-1}(n)}].$$

This formula is given by Hoffman when $[\cdot]$ is commutative (see Lemma 5.2 in [10]).

We conclude this section by an interesting formula. Let V be a vector space with basis $\{e_i\}_{i \in \mathbb{N}}$, and (V, m, σ) be the braided algebra structure given in Example 2.3. So we have that $\sigma(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$ and $m(e_i \otimes e_j) = e_{i+j}$.

For fixed i , we denote $(0)_{q_{ii}} = 1$ and $(n)_{q_{ii}} = 1 + q_{ii} + q_{ii}^2 + \dots + q_{ii}^{n-1} = \frac{1-q_{ii}^n}{1-q_{ii}}$ when $n \in \mathbb{N}$. We denote $(n)_{q_{ii}}! = (1)_{q_{ii}} \dots (n)_{q_{ii}}$.

Proposition 3.8. *For any $i, k \in \mathbb{N}$, we have that*

$$e_i^{\bowtie_{\sigma} k} = \sum_{n=1}^k \sum_{\substack{l_1 + \dots + l_n = k \\ 1 \leq l_1, \dots, l_n \leq k}} \frac{(k)_{q_{ii}}!}{(l_1)_{q_{ii}}! \dots (l_n)_{q_{ii}}!} e_{l_1 i} \otimes \dots \otimes e_{l_n i}.$$

Proof. It is a direct verification by using induction and Theorem 3.6 or Hoffman's formula. \square

4. THE DUAL CONSTRUCTION

In this section, we give a dual construction of the quantum quasi-shuffle algebra by using the universal property of tensor algebras. First of all, we study a special coalgebra structure on $T(C)$ which is a generation of the quantized cofree coalgebra structure. For coalgebras, we adopt Sweedler's notation. That means for a coalgebra (C, Δ, ε) and any $c \in C$, we denote

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

or simply, $\Delta(c) = c_{(1)} \otimes c_{(2)}$.

To extend algebra structures to quantized case, one needs the notion of braided algebras. By contrast, in the case of coalgebras, one needs the so-called braided coalgebras.

Definition 4.1. Let $C = (C, \Delta, \varepsilon)$ be a coalgebra with coproduct Δ and counit ε , and σ be a braiding on C . We call (C, Δ, σ) a *braided coalgebra* if it satisfies the following conditions:

$$\begin{cases} (\text{id}_C \otimes \Delta)\sigma &= \sigma_1 \sigma_2 (\Delta \otimes \text{id}_C), \\ (\Delta \otimes \text{id}_C)\sigma &= \sigma_2 \sigma_1 (\text{id}_C \otimes \Delta). \end{cases}$$

In order to get new braided algebras and coalgebras from old ones, we need the proposition below.

Proposition 4.2 ([8], Proposition 4.2). *1. For a braided algebra (A, μ, σ) and any $i \in \mathbb{N}$, $(A^{\otimes i}, \mu_{\sigma, i}, \beta_{ii})$ becomes a braided algebra with product $\mu_{\sigma, i} = \mu^{\otimes i} \circ T_{w_i}^\sigma$, where $w_i \in \mathfrak{S}_{2i}$ is given by*

$$w_i = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & i+1 & i+2 & \cdots & 2i \\ 1 & 3 & 5 & \cdots & 2i-1 & 2 & 4 & \cdots & 2i \end{pmatrix}.$$

2. For a braided coalgebra (C, Δ, σ) , $(C^{\otimes i}, \Delta_{\sigma, i}, \beta_{ii})$ becomes a braided coalgebra with coproduct $\Delta_{\sigma, i} = T_{w_i}^\sigma \circ \Delta^{\otimes i}$ and counit $\varepsilon^{\otimes i} : C^{\otimes i} \rightarrow \mathbb{K}^{\otimes i} \simeq \mathbb{K}$.

Let (C, Δ, σ) be a braided coalgebra. Consider the tensor algebra $T(C)$ with the concatenation product m . Then by the above proposition, $\mathcal{T}_\beta^2(C) = (T(C) \underline{\otimes} T(C), m_{\beta, 2})$ and $\mathcal{T}_\beta^3 = (T(C) \underline{\otimes} T(C) \underline{\otimes} T(C), m_{\beta, 3})$ are associative algebras.

We define

$$\begin{aligned} \phi_1 : C &\rightarrow T(C) \underline{\otimes} T(C), \\ c &\mapsto 1 \underline{\otimes} c + c \underline{\otimes} 1. \end{aligned}$$

By the universal property of tensor algebras, there exists an algebra map $\Phi_1 : T(C) \rightarrow \mathcal{T}_\beta^2(C)$ whose restriction on C is ϕ_1 . Moreover Φ_1 is coassociative and is the dual of quantum shuffle product (see, e.g., [5]).

Now we define

$$\begin{aligned} \phi_2 : C &\rightarrow T(C) \underline{\otimes} T(C), \\ c &\mapsto \sum c_{(1)} \underline{\otimes} c_{(2)}. \end{aligned}$$

By the universal property of tensor algebras again, there exists an algebra map $\Phi_2 : T(C) \rightarrow \mathcal{T}_\beta^2(C)$ whose restriction on C is ϕ_2 .

Proposition 4.3. *For any $i \in \mathbb{N}$, we have that $\Phi_2|_{C^{\otimes i}} = \Delta_{\sigma, i}$. So $(T(C), \Phi_2, \beta)$ is a braided coalgebra.*

Proof. We use induction on i . When $i = 1$, it is trivial. We assume the equality holds for the case $i < n$. Then for any $c_1, \dots, c_n \in C$, we have

$$\begin{aligned} &\Phi_2(c_1 \otimes \cdots \otimes c_n) \\ &= m_{\beta, 2}(\Phi_2(c_1) \otimes \Phi_2(c_2 \otimes \cdots \otimes c_n)) \\ &= (\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Delta(c_1) \otimes T_{w_{n-1}}^\sigma \circ \Delta^{\otimes n-1}(c_2 \otimes \cdots \otimes c_n)) \\ &= (\text{id}_C \otimes T_{\chi_{1, n-1}}^\sigma \otimes \text{id}_C^{\otimes n-1})(\text{id}_C^{\otimes 2} \otimes T_{w_{n-1}}^\sigma) \Delta^{\otimes n}(c_1 \otimes \cdots \otimes c_n) \\ &= T_{w_n}^\sigma \circ \Delta^{\otimes n}(c_2 \otimes \cdots \otimes c_n), \end{aligned}$$

where the last equality follows from the fact that $(1_{\mathfrak{S}_1} \times \chi_{1, n-1} \times 1_{\mathfrak{S}_{n-1}})(1_{\mathfrak{S}_2} \times w_{n-1}^{-1}) = w_n^{-1}$ and the expression is reduced. \square

Let $\phi = \phi_1 + \phi_2 : C \rightarrow \mathcal{T}_\beta^2(C)$ and Φ be the algebraic map induced by the universal property of tensor algebras which extends ϕ .

Proposition 4.4. *Under the notation above, the triple $(T(C), \Phi, \beta)$ is a braided coalgebra.*

Proof. We first show that

$$\begin{cases} \beta_1\beta_2(\Phi \otimes \text{id}_{T(C)}) &= (\text{id}_{T(C)} \otimes \Phi)\beta, \\ \beta_2\beta_1(\text{id}_{T(C)} \otimes \Phi) &= (\Phi \otimes \text{id}_{T(C)})\beta. \end{cases}$$

For any $x \in C^{\otimes i}$ and $y \in C^{\otimes j}$ we verify the first one on $x \underline{\otimes} y$. The second one can be verified similarly. We use induction on i .

When $i = 1$,

$$\begin{aligned} \beta_1\beta_2(\Phi \otimes \text{id}_{T(C)})(x \underline{\otimes} y) &= \beta_1\beta_2(\phi_1 \otimes \text{id} + \phi_2 \otimes \text{id})(x \underline{\otimes} y) \\ &= (\text{id} \otimes \phi_1 + \text{id} \otimes \phi_2)\beta(x \underline{\otimes} y) \\ &= (\text{id}_{T(C)} \otimes \Phi)\beta(x \underline{\otimes} y). \end{aligned}$$

For any $c \in C$, we have

$$\begin{aligned} \beta_1\beta_2(\Phi \otimes \text{id}_{T(C)})((c \otimes x) \underline{\otimes} y) &= \beta_1\beta_2\beta_3\beta_4\beta_2(\Phi \otimes \Phi \otimes \text{id}_{T(C)})(v \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_1\beta_3\beta_2\beta_3\beta_4(\Phi \otimes \Phi \otimes \text{id}_{T(C)})(c \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_1\beta_3\beta_2(\Phi \otimes \beta_1\beta_2(\Phi \otimes \text{id}_{T(C)}))(c \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_1\beta_3\beta_2(\Phi \otimes \text{id}_{T(C)} \otimes \Phi)\beta_2(v \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_3(\beta_1\beta_2(\Phi \otimes \text{id}_{T(C)}) \otimes \Phi)\beta_2(c \underline{\otimes} x \underline{\otimes} y) \\ &= \beta_3(\text{id}_{T(C)} \otimes \Phi \otimes \Phi)\beta_1\beta_2(c \underline{\otimes} x \underline{\otimes} y) \\ &= (\text{id}_{T(C)} \otimes \Phi)\beta((c \otimes x) \underline{\otimes} y). \end{aligned}$$

The next step is to show $(\Phi \otimes \text{id}_{T(C)})\Phi = (\text{id}_{T(C)} \otimes \Phi)\Phi$. Notice that for any $c \in C$,

$$\begin{aligned} &(\Phi \otimes \text{id}_{T(C)})\Phi(c) \\ &= (\Phi \otimes \text{id}_{T(C)})(c_{(1)} \underline{\otimes} c_{(2)} + 1 \underline{\otimes} c + c \underline{\otimes} 1) \\ &= c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} c_{(3)} + 1 \underline{\otimes} c_{(1)} \underline{\otimes} c_{(2)} + c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} 1 \\ &\quad + c_{(1)} \underline{\otimes} 1 \underline{\otimes} c_{(2)} + 1 \underline{\otimes} 1 \underline{\otimes} c + c_{(1)} \underline{\otimes} c_{(2)} \underline{\otimes} 1 + 1 \underline{\otimes} c \underline{\otimes} 1 + c \underline{\otimes} 1 \underline{\otimes} 1 \\ &= (\text{id}_{T(C)} \otimes \Phi)\Phi(c). \end{aligned}$$

By the uniqueness of the universal property of $T(C)$, we only need to show that both $(\Phi \otimes \text{id}_{T(C)})\Phi$ and $(\text{id}_{T(C)} \otimes \Phi)\Phi$ are algebra morphisms from $T(C)$ to \mathcal{T}_β^3 . Since $\Phi : T(C) \rightarrow \mathcal{T}_\beta^2(C)$ is an algebra morphism, we have that

$$\Phi \circ m = (m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi).$$

So

$$\begin{aligned} &(\Phi \otimes \text{id}_{T(C)}) \circ \Phi \circ m \\ &= (\Phi \otimes \text{id}_{T(C)})(m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \\ &= ((\Phi \circ m) \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \end{aligned}$$

$$\begin{aligned}
&= (m \otimes m \otimes m)(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)}) \\
&\quad \circ (\Phi \otimes \Phi \otimes \text{id}_{T(C)} \otimes \text{id}_{T(C)})(\text{id}_{T(C)} \otimes \beta \otimes \text{id}_{T(C)})(\Phi \otimes \Phi) \\
&= (m \otimes m \otimes m)\beta_2 \\
&\quad \circ (\text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes (\Phi \otimes \text{id}_{T(C)})\beta \otimes \text{id}_{T(C)})((\Phi \otimes \text{id}_{T(C)})\Phi \otimes \Phi) \\
&= (m \otimes m \otimes m)\beta_2\beta_4\beta_3 \\
&\quad \circ (\text{id}_{T(C)} \otimes \text{id}_{T(C)} \otimes \text{id}_{T(V)} \otimes \Phi \otimes \text{id}_{T(C)})((\Phi \otimes \text{id}_{T(C)})\Phi \otimes \Phi) \\
&= m_{\beta,3}((\Phi \otimes \text{id}_{T(C)})\Phi \otimes (\Phi_2 \otimes \text{id}_{T(C)})\Phi).
\end{aligned}$$

It follows that $(\Phi \otimes \text{id}_{T(C)})\Phi$ is an algebra morphism. Similarly, the map $(\text{id}_{T(C)} \otimes \Phi)\Phi$ is also an algebra morphism. \square

Now we begin to study the relation between the braided coalgebra $(T(C), \Phi, \beta)$ and the quantum quasi-shuffle algebra. We show that they are dual to each other in the following sense.

Let $\langle, \rangle: V \times W \rightarrow \mathbb{K}$ and $\langle, \rangle': V' \times W' \rightarrow \mathbb{K}$ be two bilinear non-degenerate forms. For any $f \in \text{Hom}(V, V')$, the adjoint operator $\text{adj}(f) \in \text{Hom}(W', W)$ of f is defined to be the one such that $\langle x, \text{adj}(f)(y) \rangle = \langle f(x), y \rangle'$ for any $x \in V$ and $y \in W'$. It is clear that $\text{adj}(f \circ g) = \text{adj}(g) \circ \text{adj}(f)$.

Remark 4.5. If there is a non-degenerate bilinear form between two vector spaces A and C , and (A, m, σ) is a braided algebra, then $(C, \text{adj}(m), \text{adj}(\sigma))$ is a braided coalgebra. The converse is also true.

From now on, we assume that there always exists a non-degenerate bilinear form \langle, \rangle between two vector spaces A and C . It can be extended to a bilinear form $\langle, \rangle: A^{\otimes n} \times C^{\otimes n} \rightarrow \mathbb{K}$ for any $n \geq 1$ in the usual way: for any $a_1, \dots, a_n \in A$ and $c_1, \dots, c_n \in C$,

$$\langle a_1 \otimes \dots \otimes a_n, c_1 \otimes \dots \otimes c_n \rangle = \prod_{i=1}^n \langle a_i, c_i \rangle.$$

It induces a non-degenerate bilinear form $\langle, \rangle: T(A) \times T(C) \rightarrow \mathbb{K}$ by setting that $\langle x, y \rangle = 0$ for any $x \in A^{\otimes i}$, $y \in C^{\otimes j}$ and $i \neq j$. Then we can define a non-degenerate bilinear form $\langle, \rangle: T(A) \otimes T(A) \times T(C) \otimes T(C) \rightarrow \mathbb{K}$ by requiring that $\langle u \otimes v, x \otimes y \rangle = \langle u, x \rangle \langle v, y \rangle$ for any $u, v \in T(A)$ and $x, y \in T(C)$.

If (C, Δ, σ) is a braided coalgebra, we denote $\tau = \text{adj}(\sigma)$ and $\alpha = \text{adj}(\beta)$. Then α is a braiding on $T(A)$ and $\alpha_{i,j} = T_{\chi_{ji}}^\tau = T_{\chi_{ij}}^\tau$.

Theorem 4.6. *Under the assumptions above, we have that $\text{adj}(\Phi) = \alpha_\sigma$.*

Proof. For any $c_1, \dots, c_n \in C$, we notice that

$$\begin{aligned}
\Phi(c_1 \otimes \dots \otimes c_n) &= m_{\beta,2}(\Phi(c_1) \otimes \Phi(c_2 \otimes \dots \otimes c_n)) \\
&= m_{\beta,2}((1 \otimes c_1) \otimes \Phi(c_2 \otimes \dots \otimes c_n)) \\
&\quad + m_{\beta,2}((v_1 \otimes 1) \otimes \Phi(c_2 \otimes \dots \otimes c_n)) \\
&\quad + m_{\beta,2}((c_1 \otimes 1) \otimes \Phi(c_2 \otimes \dots \otimes c_n))
\end{aligned}$$

$$\begin{aligned}
&= (\beta_{1,?} \otimes \text{id}_C^{\otimes n-1-?})(\text{id}_C \otimes \Phi)(c_1 \otimes \cdots \otimes c_n) \\
&\quad + (\text{id}_C \otimes \Phi)(c_1 \otimes \cdots \otimes c_n) \\
&\quad + (\text{id}_C \otimes \beta_{1,?} \otimes \text{id}_C^{\otimes n-1-?})(\Delta \otimes \Phi)(c_1 \otimes \cdots \otimes c_n).
\end{aligned}$$

Here, since $\Phi(C^{\otimes k}) \subset C^{\otimes 0} \underline{\otimes} C^{\otimes k} + C^{\otimes k} \underline{\otimes} C^{\otimes 0} + \cdots + C^{\otimes k} \underline{\otimes} C^{\otimes k}$, we denote by $\beta_{1,?}$ the action of β on $C \underline{\otimes} C^{\otimes ?}$ where $C^{\otimes ?}$ is the left factor of some component in $\Phi(C^{\otimes k})$.

We denote by $\text{adj}(\Phi)_{(k,l)}$ the action of $\text{adj}(\Phi)$ on $A^{\otimes k} \underline{\otimes} A^{\otimes l}$. Then on $A^{\otimes i} \underline{\otimes} A^{\otimes j}$, we have

$$\begin{aligned}
\text{adj}(\Phi)_{(i,j)} &= \text{adj}((\beta_{1,?} \otimes \text{id}_C^{\otimes i+j-1-?})(\text{id}_C \otimes \Phi) + (\text{id}_C \otimes \Phi) \\
&\quad + (\text{id}_C \otimes \beta_{1,?} \otimes \text{id}_C^{\otimes i+j-1-?})(\Delta \otimes \Phi))_{(i,j)} \\
&= (\text{id}_A \otimes \text{adj}(\Phi)_{(i,j-1)})(\alpha_{i,1} \otimes \text{id}_A^{\otimes j-1}) \\
&\quad + (\text{id}_A \otimes \text{adj}(\Phi)_{(i-1,j)}) \\
&\quad + (\text{adj}(\Delta) \otimes \text{adj}(\Phi)_{(i-1,j-1)})(\text{id}_A \otimes \beta_{i-1,1} \otimes \text{id}_A^{\otimes j-1}).
\end{aligned}$$

This shows that the map $\text{adj}(\Phi)$ shares the same inductive formula with the quantum quasi-shuffle product built on $(A, \text{adj}(\Delta), \tau)$. Hence we have the conclusion. \square

5. APPLICATIONS

In the past decade, the connection between Rota-Baxter algebras and quasi-shuffles was discovered and well studied (see [7], [4] and the references therein). Quasi-shuffle algebras built on commutative algebras provide free objects in the category of commutative Rota-Baxter algebras. Later, in [17], it is showed that quasi-shuffle algebras built on polynomial algebras are free in the category of commutative tridendriform algebras. Motivated by these works, here, we use quantum quasi-shuffle algebras to provide examples of Rota-Baxter algebras and tridendriform algebras.

We first recall the definition of Rota-Baxter algebras. For more information, one can see [7].

Definition 5.1. Let λ be an element in \mathbb{K} . A pair (R, P) is called a *Rota-Baxter algebra of weight λ* if R is a \mathbb{K} -algebra and P is a linear endomorphism of R satisfying that for any $x, y \in R$,

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy).$$

Let $(A, m, 1_A, \sigma)$ be a unital braided algebra. Then apparently $(A, \lambda \cdot m, \sigma)$ is a braided algebra for any $\lambda \in \mathbb{K}$. We denote by $\bowtie_{\sigma, \lambda}$ the quantum quasi-shuffle product with respect to $(A, \lambda \cdot m, \sigma)$. By Lemma 3 in [2], the space $A \otimes T_{\sigma}^{qsh}(A)$ is a unital associative algebra with the product

$$\diamond_{\sigma, \lambda} = (m \otimes \bowtie_{\sigma, \lambda})(\text{id}_A \otimes \beta \otimes \text{id}_{T(A)}).$$

We denote by $\mathcal{R}_{\sigma, \lambda}(A)$ the pair $(A \otimes T_{\sigma}^{qsh}(A), \diamond_{\sigma, \lambda})$. We can view $\mathcal{R}_{\sigma, \lambda}(A)$ as $T^+(A) = \oplus_{i \geq 1} A^{\otimes i}$ at the level of vector spaces. Recently, we have two products

$\bowtie_{\sigma,\lambda}$ and $\diamond_{\sigma,\lambda}$ on $T^+(A)$. This is an example of 2-braided algebras which produce quantum multi-brace algebras (for the definitions, see [12]). We define an endomorphism $P_A : \mathcal{R}_{\sigma,\lambda}(A) \rightarrow \mathcal{R}_{\sigma,\lambda}(A)$ by

$$\begin{aligned} P_A(a_0 \underline{\otimes} u) &= 1_A \underline{\otimes} a_0 \otimes u, \text{ if } u \in T^+(A), \\ P_A(a_0 \underline{\otimes} \nu) &= 1_A \underline{\otimes} \nu \cdot a_0, \text{ if } \nu \in \mathbb{K}. \end{aligned}$$

Theorem 5.2. *Under the assumptions above, the pair $(\mathcal{R}_{\sigma,\lambda}(A), P_A)$ is a Rota-Baxter algebra of weight λ .*

Proof. Observe that $\beta(u \underline{\otimes} 1_A) = 1_A \underline{\otimes} u$ for any $u \in T(A)$. Therefore for any $a, b \in A$ and $x \in A^{\otimes i}, y \in A^{\otimes j}$, we have

$$\begin{aligned} &P((a \underline{\otimes} x) \diamond_{\sigma,\lambda} P(b \underline{\otimes} y)) \\ &= P((a \underline{\otimes} x) \diamond_{\sigma,\lambda} (1_A \underline{\otimes} b \otimes y)) = P(a \underline{\otimes} (x \bowtie_{\sigma,\lambda} (b \otimes y))) \\ &= 1_A \underline{\otimes} a \otimes (x \bowtie_{\sigma,\lambda} (b \otimes y)), \end{aligned}$$

$$\begin{aligned} &P(P(a \underline{\otimes} x) \diamond_{\sigma,\lambda} (b \underline{\otimes} y)) \\ &= P((1_A \underline{\otimes} a \otimes x) \diamond_{\sigma,\lambda} (b \underline{\otimes} y)) \\ &= 1_A \underline{\otimes} ((\text{id}_A \otimes \bowtie_{\sigma,\lambda,(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y)), \end{aligned}$$

and

$$\begin{aligned} &\lambda P((a \underline{\otimes} x) \diamond_{\sigma,\lambda} (b \underline{\otimes} y)) \\ &= 1_A \underline{\otimes} ((\lambda \cdot m \otimes \bowtie_{\sigma,\lambda})(\text{id}_A \otimes \beta_{i1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y)). \end{aligned}$$

By taking a summation, we get

$$\begin{aligned} &P((a \underline{\otimes} x) \diamond_{\sigma,\lambda} P(b \underline{\otimes} y)) + P(P(a \underline{\otimes} x) \diamond_{\sigma,\lambda} (b \underline{\otimes} y)) + \lambda P((a \underline{\otimes} x) \diamond_{\sigma,\lambda} (b \underline{\otimes} y)) \\ &= 1_A \underline{\otimes} (a \otimes (x \bowtie_{\sigma,\lambda} (b \otimes y)) \\ &\quad + (\text{id}_A \otimes \bowtie_{\sigma,\lambda,(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y) \\ &\quad + (\lambda \cdot m \otimes \bowtie_{\sigma,\lambda})(\text{id}_A \otimes \beta_{i1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y)) \\ &= 1_A \underline{\otimes} ((a \otimes x) \bowtie_{\sigma,\lambda} (b \otimes y)) \\ &= P_A(a \underline{\otimes} x) \diamond_{\sigma,\lambda} P_A(b \underline{\otimes} y). \end{aligned}$$

□

Definition 5.3. A triple (R, P, σ) is called a *braided Rota-Baxter algebra of weight λ* if (R, σ) is a braided algebra and P is an endomorphism of R such that (R, P) is a Rota-Baxter algebra of weight λ and $\sigma(P \otimes P) = (P \otimes P)\sigma$.

Example 5.4. Let $(A, m, 1_A, \sigma)$ be a unital braided algebra and $(\mathcal{R}_{\sigma,\lambda}(A), P_A)$ be the Rota-Baxter algebra defined before. Then $(\mathcal{R}_{\sigma,\lambda}(A), P_A, \beta)$ is a braided Rota-Baxter algebra of weight λ .

Indeed, the only thing we need to verify is that $\beta(P_A \otimes P_A) = (P_A \otimes P_A)\beta$. For any $a, b \in A$ and $x \in A^{\otimes i}, y \in A^{\otimes j}$, we have

$$\begin{aligned}
& \beta(P_A \otimes P_A)((a \underline{\otimes} x) \underline{\otimes} (b \underline{\otimes} y)) \\
&= \beta\left((1_A \underline{\otimes} (a \otimes x)) \underline{\otimes} (1_A \underline{\otimes} (b \otimes y))\right) \\
&= 1_A \underline{\otimes} (\beta_{1,j+1} \otimes \text{id}_A^{\otimes i+1})\left(1_A \underline{\otimes} \beta_{i+1,j+1}((a \otimes x) \underline{\otimes} (b \otimes y))\right) \\
&= (P_A \otimes P_A)\beta((a \underline{\otimes} x) \underline{\otimes} (b \underline{\otimes} y)).
\end{aligned}$$

Proposition 5.5. *Let (R, P, σ) be a braided Rota-Baxter algebra of weight λ . We define*

$$x \star_P y = xP(y) + P(x)y + \lambda xy,$$

for any $x, y \in R$. If $\sigma(P \otimes \text{id}) = (\text{id} \otimes P)\sigma$ and $\sigma(\text{id} \otimes P) = (P \otimes \text{id})\sigma$, then (R, \star_P, P, σ) is again a braided Rota-Baxter algebra of weight λ .

Proof. It is well-known that (R, \star_P, P) is a Rota-Baxter algebra of weight λ . We denote by m the multiplication of R . Then we have

$$\begin{aligned}
& \sigma(\star_P \otimes \text{id}) \\
&= \sigma((m \otimes \text{id})(\text{id} \otimes P \otimes \text{id}) + (m \otimes \text{id})(P \otimes \text{id} \otimes \text{id}) + \lambda m \otimes \text{id}) \\
&= (\text{id} \otimes m)\sigma_1\sigma_2(\text{id} \otimes P \otimes \text{id}) + (\text{id} \otimes m)\sigma_1\sigma_2(P \otimes \text{id} \otimes \text{id}) \\
&\quad + (\text{id} \otimes \lambda m)\sigma_1\sigma_2 \\
&= ((\text{id} \otimes m)(\text{id} \otimes \text{id} \otimes P) + (\text{id} \otimes m)(\text{id} \otimes P \otimes \text{id}) + (\text{id} \otimes \lambda m))\sigma_1\sigma_2 \\
&= (\text{id} \otimes \star_P)\sigma_1\sigma_2.
\end{aligned}$$

The another condition $\sigma(\text{id} \otimes \star_P) = (\star_P \otimes \text{id})\sigma_2\sigma_1$ can be verified similarly. \square

Given any braided Rota-Baxter algebra (R, P, σ) , the above proposition provides another example of 2-braided algebra.

Now we turn to tridendriform algebras which were introduced by Loday and Ronco ([18]).

Definition 5.6. Let V be a vector space, and \prec, \succ and \cdot be three binary operations on V . The quadruple (V, \prec, \succ, \cdot) is called a *tridendriform algebra* if the following relations are satisfied: for any $x, y, z \in V$,

$$\begin{aligned}
(x \prec y) \prec z &= x \prec (y \ast z), \\
(x \succ y) \prec z &= x \succ (y \prec z), \\
(x \ast y) \succ z &= x \succ (y \succ z), \\
(x \succ y) \cdot z &= x \succ (y \cdot z), \\
(x \prec y) \cdot z &= x \cdot (y \succ z), \\
(x \cdot y) \prec z &= x \cdot (y \prec z), \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z),
\end{aligned}$$

where $x \ast y = x \prec y + x \succ y + x \cdot y$.

Theorem 5.7. *Let (A, m, σ) be a braided algebra. We define three operations \cdot, \prec and \succ on $T(A)$ recursively by: for any $a, b \in A$ and any $x \in A^{\otimes i}, x \in A^{\otimes i}$,*

$$\begin{aligned} (a \otimes x) \cdot (b \otimes y) &= (m \otimes \bowtie_{\sigma(i,j)})(\text{id}_A \otimes \beta_{i,1} \otimes \text{id}_A^{\otimes j-1})(a \otimes x \otimes b \otimes y), \\ (a \otimes x) \prec (b \otimes y) &= a \otimes (x \bowtie_{\sigma} (b \otimes y)), \\ (a \otimes x) \succ (b \otimes y) &= (\text{id}_A \otimes \bowtie_{\sigma(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_A^{\otimes j})(a \otimes x \otimes b \otimes y). \end{aligned}$$

Then $(T(A), \prec, \succ, \cdot)$ is a tridendriform algebra.

Proof. All the verifications are direct. We just need that \bowtie_{σ} is associative and compatible with the braiding β in the sense of braided algebras. For instance, we show the third condition. For any $a \in A$ and $x, y, z \in T(A)$,

$$\begin{aligned} (x \bowtie_{\sigma} y) \succ (a \otimes z) &= (\text{id}_A \otimes \bowtie_{\sigma})(\beta_{?,1} \otimes \text{id}_{T(A)})(\bowtie_{\sigma} \otimes \text{id}_A \otimes \text{id}_{T(A)})(x \otimes y \otimes a \otimes z) \\ &= (\text{id}_A \otimes \bowtie_{\sigma})(\beta_{?,1}(\bowtie_{\sigma} \otimes \text{id}_A) \otimes \text{id}_{T(A)})(x \otimes y \otimes a \otimes z) \\ &= (\text{id}_A \otimes \bowtie_{\sigma})(\text{id}_A \otimes \bowtie_{\sigma} \otimes \text{id}_{T(A)}) \\ &\quad \circ (\beta_{?,1} \otimes \text{id}_{T(A)} \otimes \text{id}_{T(A)})(\text{id}_{T(A)} \otimes \beta_{?,1} \otimes \text{id}_{T(A)})(x \otimes y \otimes a \otimes z) \\ &= (\text{id}_A \otimes \bowtie_{\sigma})(\text{id}_A \otimes \text{id}_{T(A)} \otimes \bowtie_{\sigma}) \\ &\quad \circ (\beta_{?,1} \otimes \text{id}_{T(A)} \otimes \text{id}_{T(A)})(\text{id}_{T(A)} \otimes \beta_{?,1} \otimes \text{id}_{T(A)})(x \otimes y \otimes a \otimes z) \\ &= (\text{id}_A \otimes \bowtie_{\sigma})(\beta_{?,1} \otimes \text{id}_{T(A)}) \\ &\quad \circ (\text{id}_{T(A)} \otimes \text{id}_A \otimes \bowtie_{\sigma})(\text{id}_{T(A)} \otimes \beta_{?,1} \otimes \text{id}_{T(A)})(x \otimes y \otimes a \otimes z) \\ &= x \succ (y \succ (a \otimes z)). \end{aligned}$$

□

Remark 5.8. Let (R, \cdot, P) be a Rota-Baxter algebra of weight 1. Define $a \prec b = a \cdot P(b)$ and $a \succ b = P(a) \cdot b$. Then (R, \prec, \succ, \cdot) is a tridendriform algebra (see [17]). Using this fact, we can prove the above theorem by a more easier argument: embed A into the unital braided algebra $(\tilde{A}, \tilde{m}, 1, \tilde{\sigma})$, then the tridendriform algebra structure in Theorem 5.8 comes from the Rota-Baxter algebra $(\mathcal{R}_{\tilde{\sigma},1}(\tilde{A}), P_{\tilde{A}})$.

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